The Channel Assignment Problem in 3-Dimensions

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1 Introduction

In a wireless network, every wireless device communicates to transmit data. In a cellular network, wireless devices communicate with base stations to transmit data. Devices on a wireless network communicate with other devices on certain frequencies. When two devices that are close to each other communicate on frequencies that do not differ by much, there is interference, leading to transmission loss. Because of the increasing density of devices and the limited bandwidth of frequencies that providers are alloted, it is desirable to assign frequency channels to base stations so that:

1. frequencies assigned to nearby base stations are far enough apart to assure that there will be no interference, and
2. the number of frequencies used across the network of base stations is minimized.

The distance of separation where a frequency can be reused is referred to as the *co-channel reuse distance* and is denoted by σ. Given a particular network, finding a scheme to assign frequencies, also referred to as channels, to the base stations is referred to as the channel assignment problem. Most current research on this problem assumes that the networks work in a 2-dimensional environment [1]. Actual networks of this type are generally in 3-dimensions, as in the case of a multi-floored building where the different levels have their own access points. Also, an increasing number of service providers operate in the same geographical area. These service providers have to operate on frequencies that are different from one another. This situation can be modeled in 3-dimensions. If we model each service provider's network as a 2-dimensional grid, and each one is "stacked" on top of one another, we can form a 3-dimensional grid. These two cases sparked the interest in studying the channel assignment problem in 3-dimensions.

2 Previous Research

In the channel assignment problem, we attempt to assign frequencies $0, 1, 2, \ldots$ to the vertices of the square grid such that two points within a given distance σ (where the distance is the number of edges between the two points) do not get assigned the same frequency. This problem can be abstractly viewed as the problem of coloring the network graph using colors $0, 1, 2, \ldots$. The number of frequencies (colors) used in such a coloring is referred to as the *span* of the coloring. If such a span is the smallest possible, then the coloring of the network is said to as be *optimal*.

Now consider the largest subgraph of a given graph such that any 2 points in said subgraph are at a distance less than σ. In this graph, each vertex must have its own unique color. In [1], they developed an algorithm to calculate the size, denoted by $n(\sigma, d)$, of Z^d of such a subgraph, which is represented as $n(\sigma, d)$:

$$n(\sigma, d) = \begin{cases} \sigma, & \text{If } d = 1; \\ n(\sigma, d-1) + 2\sum_{i=1}^{\lfloor \frac{\sigma}{2} \rfloor} n(\sigma - 2i, d-1), & \text{otherwise.} \end{cases}$$

In [1], they also developed the notion of a coloring schema that can be used to color the graph Z^d (which represents the d-dimensional square grid) for all values of d and odd values of σ. The following definition for a coloring schema comes from [1]:

Definition 1. *For $d \geq 1$, suppose $\sigma > 1, N \geq n(\sigma, d)$ are odd integers, and $T = \langle t_1, t_2, \ldots, t_{d-1} \rangle$ is a non-decreasing sequence of $(d-1)$ positive, odd integers. Then (σ, T, N) is a* colouring schema for $\mathbf{Z}^d$*, denoted $\mathcal{S}_d$, iff*

1. *for each i, $1 \leq i < d$, $\sigma \leq t_i \leq N$, and*
2. *for all $X = (x_0, \ldots, x_{d-1}) \in \mathbf{Z}^d$, $X \neq \vec{0}$,*

$$\sum_{i=0}^{d-1} |x_i| < \sigma \Longrightarrow \left(x_0 + \sum_{i=1}^{d-1} x_i \cdot t_i \right) \textit{ mod } N \neq 0.$$

Also in [1], they defined an algorithm to color the graph Z^d given a coloring schema.

3 A Family of Coloring Schema for Z^3

Theorem 1. *$\forall\, \sigma \geq 3$, $\forall\, t_1 \geq \sigma$, $T =< t_1, t_2 = (\sigma-1)t_1+1 >$ and $N = (\sigma-1)\times t_2+1$ is a coloring schema for $d = 3$.*

Proof. To be a coloring schema, conditions one and two in Definition 1 must be satisfied:

1. t_1 and $t_2 \geq \sigma$.

   From the premise of Theorem 1, $t_1 \geq \sigma$.

   $t_2 = (\sigma-1)t_1 + 1 \geq (\sigma-1)\sigma + 1 > \sigma$.

2. $N \geq n(\sigma, 3)$

   The general form for $n(\sigma, 3) = \frac{4}{3}k^3 + 2k^2 + \frac{8}{3}k + 1$, for $k = \frac{\sigma-1}{2}$. To show that $N \geq n(\sigma, 3)$, we must show that $N - n(\sigma, 3) > 0$.

$$\begin{aligned} N &= (\sigma-1)\times t_2 + 1 \\ &= (\sigma-1)\times((\sigma-1)\times t_1 + 1) + 1 \\ &\geq (\sigma-1)\times((\sigma-1)\times\sigma + 1) + 1 \\ &= \sigma^3 - 2\sigma^2 + 2\sigma \\ &= 8k^3 + 4k^2 + 2k + 1, \text{ since } \sigma = 2k+1. \end{aligned}$$

   Then, for all $k > 0$, i.e., $\sigma \geq 3$,

$$8k^3 + 4k^2 + 2k + 1 - \frac{4}{3}k^3 - 2k^2 - \frac{8}{3}k - 1 = \frac{20}{3}k^3 + 2k^2 - \frac{2}{3}k > 0.$$

3. $\forall\vec{x}$, $\vec{x} \neq \vec{0}$, and $\sum|x_i| < \sigma$, $x_0 + x_1\times t_1 + x_2\times t_2 \neq 0(\bmod N)$.

We know that $\forall \vec{x}$, where $\sum |x_i| < \sigma$, $x_0 + x_1 \times t_1 + x_2 \times t_2 < N$. We know this because the biggest that $x_0 + x_1 \times t_1 + x_2 \times t_2$ will be is when $x_2 = (\sigma - 1)$. This would make $x_0 + x_1 \times t_1 + x_2 \times t_2 \leq x_0 + x_1 t_1 + (\sigma - 1)t_2$. Then, since $x_0 = x_1 = 0$, $x_0 + x_1 t_1 + x_2 t_2 \leq (\sigma - 1)t_2 = (\sigma - 1)(\sigma(\sigma - 1) + 1) = \sigma^3 - 2\sigma^2 + 2\sigma - 1 < N$. Therefore, all we need to show is: $\forall \vec{x} \neq \vec{0}$, $\sum |x_i| < \sigma$, $x_0 + x_1 \times t_1 + x_2 \times t_2 \neq 0$.

(a) Case 1: $x_2 = 0$.

In this case, in order for $x_0 + x_1 \times t_1 + x_2 \times t_2 = 0$, $x_0 + x_1 \times t_1 = 0$. We know that $t_1 \geq \sigma$. If $x_1 = 0$, then x_0 would have to be 0 as well, Thus $\vec{x} = \vec{0}$, giving us a contradiction. If $x_1 \neq 0$, then $|x_1 t_1| \geq \sigma$, since $t_1 \geq \sigma$ and $|x_1| \geq 1$. So, in order for the sum to be 0, $|x_0| \geq \sigma$, giving us a contradiction.

(b) Case 2: $x_2 \neq 0$.

In this case, in order for $x_0 + x_1 \times t_1 + x_2 \times t_2 = 0$, $|x_2 t_2| - |x_1 t_1| - |x_0| = 0$. Let $x_2 = 1$, this means that $|t_2| - |x_1 t_1| - |x_0| = 0$. Since $t_2 = (\sigma - 1) \times t_1 + 1$, $x_1 = \sigma - 1$, but this would cause $\sum |x_i| \geq \sigma$, which causes a contradiction.

Thus, $x_0 + x_1 \times t_1 + x_2 \times t_2 \neq 0 (mod N)$, $\forall \vec{x}$, where $\sum |x_i| < \sigma$.

Therefore, since all of the conditions were met, we have a coloring schema for $d = 3$. □

In the above proof, the span of the coloring was N. This N is greater than the optimal span $n(\sigma, 3)$. But, since N and $n(\sigma, d)$ have the same order, they are growing at the same rate. Also, using the results from [1], our coloring schema yields a coloring with $\delta_1 = \frac{\sigma^3 - 3\sigma^2 + 3\sigma - 1}{2}$.

4 Empirical Data for Minimizing Coloring Span

In the previous section, we noted that our value for N was not optimal. We wished to minimize the span of the coloring. The following table, calculated through exhaustive search, lists values of $\sigma, N, t_1 = \sigma$ and t_2 where $N > n(\sigma, 3)$ is the smallest number admitting a coloring schema as witnessed by t_2.

σ	$n(\sigma, 3)$	N	t_1	t_2	σ	$n(\sigma, 3)$	N	t_1	t_2
3	7	7	3	5	27	3303	3751	27	561
5	25	27	5	8	29	4089	4635	29	631
7	63	71	7	30	31	4991	5663	31	737
9	129	145	9	61	33	6017	6817	33	817
11	231	263	11	97	35	7175	8135	35	937
13	377	427	13	127	37	8473	9595	37	1027
15	575	655	15	177	39	9919	11239	39	1161
17	833	945	17	217	41	11521	13041	41	1261
19	1159	1391	19	281	43	13287	15047	43	1409
21	1561	1771	21	331	45	15225	17227	45	1519
23	2047	2327	23	409	47	17343	19631	47	1681
25	2625	2977	25	469	49				

Also in our research, we developed several checks in hopes of reducing the time required to exhaustively search for these coloring schema. Of the developed checks, only one seemed to help. In the checks, if the condition is satisfied, we can throw out the values of t_1 and t_2:

1. $|t_2 - (i \times t_2)| < \sigma - (i+1)$ because the point $(\sigma - (i+1), i, 1)$ will cause the values to fail.

5 Conclusion

In this paper, we have shown a family of coloring schema for Z^3. We have also provided empirical data on values that reduce the span of the coloring of a graph. It may be interesting to see if there could be some closed form for the values of N that were generated.

It is possible to extend the family of coloring schema, described in Section 3, to dimensions higher than 3. Further research could find other such families of coloring in these higher dimensions.

Lastly, networks of this type are usually modeled using a cellular, triangular grid. It may be of interest to see how using such a grid could affect the graph colorings in higher dimensions as well.

References

[1] Aniket Dubhashi, Shashanka MVS, Amrita Pati, Shashank R., and Anil M. Shende. Channel assignment for wireless networks modelled as d-dimensional square grids. In *IWDC '02: Proceedings of the 4th International Workshop on Distributed Computing, Mobile and Wireless Computing*, pages 130–141, London, UK, 2002. Springer-Verlag.